Approximation of Bivariate Functions from Fractional Sobolev Spaces by Filtered Back Projection

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Abstract

We analyse the convergence of filtered back projection methods to approximate bivariate functions from fractional Sobolev spaces. To this end, we prove estimates for the inherent approximation error that is incurred by the usage of a low-pass filter. This yields error estimates for a scale of rougher Sobolev spaces, which also covers the L^2-case. In this way, we extend previous of our results concerning L^2-error estimates to Sobolev error estimates, but now under weaker conditions on the window function of the utilized low-pass filter. In fact, we only require bounded windows, unlike in our previous work, where we relied on continuous windows. Therefore, the analysis of this paper now applies to a larger class of commonly used low-pass filters, including the Ram-Lak and the Shepp-Logan filter. Finally, we prove asymptotic convergence rates for the FBP approximation error, as the window's bandwidth tends to infinity, where we show that the resulting decay rate is given by the difference between the smoothness of the target function and the order of the rougher Sobolev space.

Keywords: Filtered back projection, fractional Sobolev spaces, error bounds, asymptotic convergence rates.

2010 MSC: 41A25, 94A20, 94A08

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1. Introduction

The method of *filtered back projection* (FBP) is a well-known reconstruction technique in computerized tomography (CT) [9], which is commonly used in relevant applications, e.g. in medical imaging [4] or in non-destructive evaluation of materials.

The main purpose of FBP in CT is to recover the interior structure of an unknown object from X-ray scans, where the measured X-ray data is interpreted as a set of line integrals for a bivariate function, termed *attenuation function*. We can formulate the mathematical problem in the recovery step of FBP as follows.

Problem 1.1. *On given domain $\Omega \subset \mathbb{R}^2$, reconstruct a bivariate function $f \in \mathrm{L}^1(\Omega)$ from its line integrals*

$$\int_{\ell} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

which are assumed to be given for all straight lines $\ell \subset \mathbb{R}^2$ passing through Ω. ■

To parametrise straight lines in the plane, let $\ell_{t,\theta} \subset \mathbb{R}^2$ denote the unique straight line which passes through $(t\cos(\theta), t\sin(\theta)) \in \mathbb{R}^2$, for $(t,\theta) \in \mathbb{R} \times [0,\pi)$, and which is perpendicular to the unit vector $\vec{n}_\theta = (\cos(\theta), \sin(\theta))$. In this way, any straight line $\ell \equiv \ell_{t,\theta} \subset \mathbb{R}^2$ can be represented by unique parameters $(t,\theta) \in \mathbb{R} \times [0,\pi)$. This leads us to the *Radon transform* $\mathscr{R}$, defined as

$$(\mathscr{R}f)(t,\theta) = \int_{\ell_{t,\theta}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{for } (t,\theta) \in \mathbb{R} \times [0,\pi).$$

Note that the Radon transform $\mathscr{R}$ is a linear integral operator that maps a bivariate function $f \equiv f(x,y) \in \mathrm{L}^1(\mathbb{R}^2)$ in Cartesian coordinates onto a bivariate function $\mathscr{R}f \equiv (\mathscr{R}f)(t,\theta)$ in polar coordinates. For a comprehensive mathematical treatment of the Radon transform, we refer to [5, 9].

Therefore, the basic reconstruction problem, Problem 1.1, seeks for the inversion of the Radon transform $\mathscr{R}f$ from input Radon data $\left\{\mathscr{R}f(t,\theta) \mid t \in \mathbb{R}, \theta \in [0,\pi)\right\}$. We remark that this problem has a very long history, dating back to Johann Radon, whose seminal work [10] provided an analytical inversion of $\mathscr{R}$ already in 1917. This has later led to the *filtered back projection* (FBP) formula (see [4, 9]),

$$f(x,y) = \frac{1}{2}\mathscr{B}\left(\mathscr{F}^{-1}[|S|\mathscr{F}(\mathscr{R}f)(S,\theta)]\right)(x,y) \quad \text{for all } (x,y) \in \mathbb{R}^2, \tag{1}$$

where the back projection $\mathscr{B}$ is the adjoint operator of $\mathscr{R}$, and where $\mathscr{F}$ is the univariate Fourier transform acting on variable S. In the following of this paper, we explain on (1) and its ingredients in more detail. For the moment, we only wish to remark that the FBP formula (1) is highly sensitive with respect to noise, due to the *filter* $|S|$.

In standard stabilization methods for the FBP formula, $|S|$ in (1) is replaced by a compactly supported *low-pass filter* $A_L : \mathbb{R} \longrightarrow \mathbb{R}$, with bandwidth $L > 0$, of the form

$$A_L(S) = |S|\, W(S/L), \tag{2}$$

where $W \in \mathrm{L}^\infty(\mathbb{R})$ is an even *window* function of compact support $\mathrm{supp}(W) \subseteq [-1,1]$, so that $\mathrm{supp}(A_L) \subseteq [-L,L]$. This modification leads us to an *approximate* FBP formula

$$f_L(x,y) = \frac{1}{2} \mathscr{B}\left(\mathscr{F}^{-1}[A_L(S)\mathscr{F}(\mathscr{R}f)(S,\theta)]\right)(x,y). \tag{3}$$

Examples for windows W of commonly used low-pass filters $A_L(S) = |S|\, W(S/L)$ are shown in Table 1.

Table 1: Window functions of commonly used low-pass filters.

Name	$W(S)$ for $\lvert S\rvert \leq 1$	Parameter
Ram-Lak	1	-
Shepp-Logan	$\mathrm{sinc}(\pi S/2)$	-
Cosine	$\cos(\pi S/2)$	-
Hamming	$\beta + (1-\beta)\cos(\pi S)$	$\beta \in [1/2, 1]$
Gaussian	$\exp\left(-(\pi S/\beta)^2\right)$	$\beta > 1$

In this paper, we analyse the *inherent* reconstruction error

$$e_L = f - f_L \tag{4}$$

of the FBP approximation f_L that is incurred by the chosen low-pass filter A_L, for $L > 0$.

We remark that pointwise and L^∞-error estimates on e_L in (4) were proven by Munshi et al. in [7]. Their results are further supported by numerical experiments in [8]. Error bounds on the L^p-norm of e_L, in terms of an L^p-modulus of continuity of the target function f, were proven by Madych in [6]. But our approach is essentially different from previous approaches, in particular different from that in [6].

We prove error estimates on e_L in (4) for target functions f from Sobolev spaces $\mathrm{H}^\alpha(\mathbb{R}^2)$ of fractional order $\alpha > 0$. Recall that the *Sobolev space* $\mathrm{H}^\alpha(\mathbb{R}^2)$ of order α is defined as

$$\mathrm{H}^\alpha(\mathbb{R}^2) = \big\{ f \in \mathscr{S}'(\mathbb{R}^2) \mid \|f\|_\alpha < \infty \big\},$$

where $\mathscr{S}'(\mathbb{R}^2)$ consists of the tempered distributions with respect to the Schwartz space $\mathscr{S}(\mathbb{R}^2)$, and where the Sobolev norm $\|\cdot\|_\alpha \equiv \|\cdot\|_{\mathrm{H}^\alpha(\mathbb{R}^2)}$ on $\mathrm{H}^\alpha(\mathbb{R}^2)$ is given as

$$\|f\|_\alpha^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \big(1 + x^2 + y^2\big)^\alpha \, |\mathscr{F} f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y \qquad \text{for } f \in \mathrm{H}^\alpha(\mathbb{R}^2).$$

In previous work [1, 2] we analysed the approximation properties of f_L in (3). More precisely, in [2, Theorem 4.1] we proved L^2-error estimates of the form

$$\|f - f_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \Big(\Phi_{\alpha,W}^{1/2}(L) + L^{-\alpha} \Big) \, \|f\|_\alpha \tag{5}$$

for target functions $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, where $\alpha > 0$, and where $\Phi_{\alpha,W} : \mathbb{R} \longrightarrow \mathbb{R}$ is a specific function that we discuss later. Moreover, in [2, Theorem 4.2] we showed the convergence of the FBP approximation, where we proved $\Phi_{\alpha,W}(L) \longrightarrow 0$ for $L \longrightarrow 0$.

To establish the L^2-error estimate in (5), along with the convergence of the FBP approximation, we were implicitly relying on the continuity of the window function W, i.e., we required $W \in \mathscr{C}(\mathbb{R})$. This assumption, is, however, too restrictive, so that commonly used low-pass filters, including the Ram-Lak and the Shepp-Logan filter, are not covered by our analysis in [1, 2].

In this paper, we show how to maintain our previous L^2-error estimates and convergence results from [1, 2], in particular that in (5), but now under weaker conditions on the window function W, where we only require $W \in \mathrm{L}^\infty(\mathbb{R})$ with $\mathrm{supp}(W) \subseteq [-1,1]$. In this way, the analysis of this paper also covers the Ram-Lak and the Shepp-Logan filter.

Moreover, we extend our error analysis from $\mathrm{L}^2(\mathbb{R}^2)$ to fractional Sobolev spaces in order to obtain asymptotic error estimates of the form

$$\|f - f_L\|_\sigma \leq \big(c \, \|1 - W\|_{\infty,[-1,1]} + 1 \big) \, L^{\sigma - \alpha} \, \|f\|_\alpha = \mathscr{O}(L^{\sigma - \alpha})$$

for $L \longrightarrow \infty$, in the rougher Sobolev spaces $\mathrm{H}^\sigma(\mathbb{R}^2)$, for $0 \leq \sigma \leq \alpha$.

In our subsequent analysis, we rely, as in [1, 2], on the representation

$$f_L = \frac{1}{2} \mathscr{B} \big(\mathscr{F}^{-1} A_L * \mathscr{R} f \big) = f * K_L, \tag{6}$$

with the *convolution kernel*

$$K_L = \frac{1}{2}\mathscr{B}(\mathscr{F}^{-1}A_L), \tag{7}$$

whose properties are entirely determined by the window W and the bandwidth $L > 0$. In [1, 2], we essentially require the assumption $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. In this paper, we show how to avoid this restriction on K_L, while maintaining the representations (6) and (7) in the L^2-sense, where, again, we only rely on the assumption $W \in \mathrm{L}^\infty(\mathbb{R})$ with $\mathrm{supp}(W) \subseteq [-1,1]$.

The outline of this paper is as follows. In Section 2, we briefly review relevant facts concerning the Radon transform. This is followed by an analysis on the properties of the FBP approximation f_L (in Section 3) and of the convolution kernel K_L (in Section 4), where we establish the identities (6) and (7) in the L^2-sense. In Section 5 we finally perform an analysis on the FBP reconstruction error e_L in (4). This includes both L^2 and Sobolev error estimates and resulting convergence rates, under rather mild assumptions on the low-pass filter's window $W \in \mathrm{L}^\infty(\mathbb{R})$.

2. Preliminaries concerning the Radon Transform

The analytical properties of the Radon transform are well-understood (see [5, 9]). In particular, the inversion of the Radon transform $\mathscr{R}$ is given by the filtered back projection (FBP) formula (1). Nevertheless, for the reader's convenience, we briefly collect only a few relevant facts concerning the Radon transform, on which we rely later in this paper. On this occasion, we introduce some basic notations. Since the following results are well-known, we omit the proofs and refer to the literature instead.

We start with the (continuous) univariate *Fourier transform* on $\mathbb{R}$, here taken as

$$\mathscr{F}g(S,\theta) = \int_{\mathbb{R}} g(t,\theta)\,\mathrm{e}^{-\mathrm{i}tS}\,\mathrm{d}t \qquad \text{for } (S,\theta) \in \mathbb{R} \times [0,\pi)$$

for $g \equiv g(t,\theta)$ satisfying $g(\cdot,\theta) \in \mathrm{L}^1(\mathbb{R})$ for all $\theta \in [0,\pi)$, and the *back projection*

$$\mathscr{B}h(x,y) = \frac{1}{\pi}\int_0^\pi h(x\cos(\theta) + y\sin(\theta),\theta)\,\mathrm{d}\theta \qquad \text{for } (x,y) \in \mathbb{R}^2$$

for $h \in \mathrm{L}^1(\mathbb{R} \times [0,\pi))$. Note that the back projection $\mathscr{B}$ maps a bivariate function $h \equiv h(t,\theta)$ in polar coordinates, satisfying $h(t,\cdot) \in \mathrm{L}^1([0,\pi))$ for any $t \in \mathbb{R}$, onto a bivariate function $\mathscr{B}h \equiv \mathscr{B}h(x,y)$ in Cartesian coordinates.

Later in this paper, we work with the bivariate *Fourier transform*, given as

$$\mathscr{F}f(X,Y) = \int_{\mathbb{R}^2} f(x,y)\,\mathrm{e}^{-\mathrm{i}(xX+yY)}\,\mathrm{d}x\,\mathrm{d}y \quad \text{for } (X,Y)\in\mathbb{R}^2,$$

for $f \equiv f(x,y)$ in Cartesian coordinates, where we assume $f \in \mathrm{L}^1(\mathbb{R}^2)$.

Now we collect basic properties about $\mathscr{R}$ that we need in our subsequent analysis. We first recall that for $f \in \mathrm{L}^1(\mathbb{R}^2)$ its Radon transform $\mathscr{R}f$ is in $\mathrm{L}^1(\mathbb{R}\times[0,\pi))$.

Lemma 2.1. *Let $f \in \mathrm{L}^1(\mathbb{R}^2)$. Then, $\mathscr{R}f \in \mathrm{L}^1(\mathbb{R}\times[0,\pi))$ with*

$$\|\mathscr{R}f\|_{\mathrm{L}^1(\mathbb{R}\times[0,\pi))} \le \pi\,\|f\|_{\mathrm{L}^1(\mathbb{R}^2)}.$$

Moreover, if f has compact support, i.e., there exists an $R>0$ satisfying

$$f(x,y) = 0 \quad \textit{for all } (x,y)\in\mathbb{R}^2 \textit{ with } \|(x,y)\|_{\mathbb{R}^2} > R,$$

then $\mathscr{R}f$ has compact support by

$$\mathscr{R}f(t,\theta) = 0 \quad \textit{for all } (t,\theta)\in\mathbb{R}\times[0,\pi) \textit{ with } |t| > R. \qquad \square$$

Next we recall that the L^2-norm of $\mathscr{R}f$ is bounded, provided that the function f belongs to $\mathrm{L}^2_0(\mathbb{R}^2)$, i.e., f is square integrable and has compact support.

Lemma 2.2. *Let $f \in \mathrm{L}^2_0(\mathbb{R}^2)$ be supported in a compact set $K \subset \mathbb{R}^2$ with diameter*

$$\mathrm{diam}(K) = \sup\{\|(x-X,y-Y)\|_{\mathbb{R}^2} \mid (x,y),(X,Y)\in K\} < \infty.$$

Then, $\mathscr{R}f \in \mathrm{L}^2(\mathbb{R}\times[0,\pi))$ with

$$\|\mathscr{R}f\|^2_{\mathrm{L}^2(\mathbb{R}\times[0,\pi))} \le \pi\,\mathrm{diam}(K)\,\|f\|^2_{\mathrm{L}^2(\mathbb{R}^2)}. \qquad \square$$

Lemma 2.2 indicates that the Radon transform $\mathscr{R}$ can be viewed as a densely defined unbounded linear operator from $\mathrm{L}^2(\mathbb{R}^2)$ to $\mathrm{L}^2(\mathbb{R}\times[0,\pi))$ with domain $\mathrm{L}^2_0(\mathbb{R}^2)$.

Finally, we turn to the adjoint operator $\mathscr{R}^\#$ of the Radon transform $\mathscr{R}$.

Lemma 2.3 (see [11, Theorem 12.3]). *The adjoint operator of the Radon transform $\mathscr{R} : \mathrm{L}^2_0(\mathbb{R}^2) \longrightarrow \mathrm{L}^2(\mathbb{R}\times[0,\pi))$ is given by*

$$\mathscr{R}^\# g(x,y) = \int_0^\pi g(x\cos(\theta)+y\sin(\theta),\theta)\,\mathrm{d}\theta \quad \textit{for } (x,y)\in\mathbb{R}^2.$$

For every $g \in \mathrm{L}^2(\mathbb{R}\times[0,\pi))$, $\mathscr{R}^\# g$ is defined almost everywhere on $\mathbb{R}^2$ and satisfies

$$\mathscr{R}^\# g \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2). \qquad \square$$

Lemma 2.3 says that the back projection $\mathscr{B}$ is, up to constant $\frac{1}{\pi}$, the adjoint operator of the Radon transform $\mathscr{R}$, i.e., $\mathscr{B} = \frac{1}{\pi}\mathscr{R}^{\#}$. In particular, for $g \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$ the function $\mathscr{B}g$ is defined almost everywhere on $\mathbb{R}^2$ and satisfies $\mathscr{B}g \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$.

3. Representation of the Filtered Back Projection Approximation

Now we return to the *approximate* filtered back projection (FBP) formula (3), whose properties are entirely determined by the low-pass filter A_L in (2), particularly by the choice of the window function $W \in \mathrm{L}^\infty(\mathbb{R})$. In this section we prove useful properties concerning the main ingredients of f_L in (3). This will later allow us to construct asymptotic Sobolev error estimates and convergence rates, where the so obtained bounds on the error $e_L = f - f_L$ will cover our previous ones in [1, 2], but under weaker conditions on the window function W.

Our first result concerns the representation

$$f_L = \frac{1}{2}\mathscr{B}\left(\mathscr{F}^{-1}A_L * \mathscr{R}f\right) \tag{8}$$

in (6), where $*$ denotes the *univariate* convolution product with respect to the radial variable, i.e., for $g \equiv g(\cdot,\theta) \in \mathrm{L}^1(\mathbb{R})$ and $h \equiv h(\cdot,\theta) \in \mathrm{L}^1(\mathbb{R})$ we let

$$(g * h)(S,\theta) = \int_{\mathbb{R}} g(t,\theta)\,h(S-t,\theta)\,\mathrm{d}t \qquad \text{for } (S,\theta) \in \mathbb{R} \times [0,\pi).$$

We now show that f_L in (3) is well-defined, for $f \in \mathrm{L}^1(\mathbb{R}^2)$ and $W \in \mathrm{L}^\infty(\mathbb{R})$, and satisfies (8). For convenience, we introduce the band-limited function $q_L : \mathbb{R} \times [0,\pi) \longrightarrow \mathbb{R}$ via

$$q_L(S,\theta) = \mathscr{F}^{-1}A_L(S) \qquad \text{for } (S,\theta) \in \mathbb{R} \times [0,\pi). \tag{9}$$

Note that q_L is well-defined on $\mathbb{R} \times [0,\pi)$, where $q_L \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$, since the low-pass filter A_L in (2) is compactly supported, so that $A_L \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$ for all $L > 0$.

Proposition 3.1. *Let* $f \in \mathrm{L}^1(\mathbb{R}^2)$. *Moreover, let* $W \in \mathrm{L}^\infty(\mathbb{R})$ *be even and compactly supported with* $\mathrm{supp}(W) \subseteq [-1,1]$. *Then,* f_L *in* (3) *is for any* $L > 0$ *defined almost everywhere (a.e.) on* $\mathbb{R}^2$ *and satisfies* $f_L \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$. *Further,* f_L *can be rewritten as*

$$f_L = \frac{1}{2}\mathscr{B}(q_L * \mathscr{R}f) \qquad \text{a.e. on } \mathbb{R}^2$$

with the band-limited function $q_L \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$ *defined in* (9).

Our proof of Proposition 3.1 relies on the *Fourier convolution theorem*.

Lemma 3.2 (see [12, Theorem I.2.6]). *For $d \in \mathbb{N}$, let $f \in \mathrm{L}^1(\mathbb{R}^d)$ and $g \in \mathrm{L}^2(\mathbb{R}^d)$. Then, $f * g$ belongs to $\mathrm{L}^2(\mathbb{R}^d)$ and the representation*

$$\mathscr{F}(f * g) = \mathscr{F} f \cdot \mathscr{F} g$$

holds in the L^2-sense, in particular almost everywhere on $\mathbb{R}^d$. □

Proof of Proposition 3.1: For $f \in \mathrm{L}^1(\mathbb{R}^2)$, we obtain $\mathscr{R}f \in \mathrm{L}^1(\mathbb{R} \times [0,\pi))$ by Lemma 2.1. Consequently, $\mathscr{F}(\mathscr{R}f)(\cdot,\theta) \in \mathscr{C}_0(\mathbb{R})$ for all $\theta \in [0,\pi)$ by the Riemann-Lebesgue lemma (see e.g. [12, Theorem I.1.2]). Since $W \in \mathrm{L}^\infty(\mathbb{R})$ has compact support, this implies that the function $(S,\theta) \longmapsto A_L(S)\mathscr{F}(\mathscr{R}f)(S,\theta)$ is in $\mathrm{L}^2(\mathbb{R} \times [0,\pi))$. Thus, by the Rayleigh-Plancherel theorem (see e.g. [12, Theorem I.2.3])

$$(S,\theta) \longmapsto \mathscr{F}^{-1}[A_L(S)\mathscr{F}(\mathscr{R}f)(S,\theta)]$$

is also in $\mathrm{L}^2(\mathbb{R} \times [0,\pi))$ and so

$$(x,y) \longmapsto \frac{1}{2}\mathscr{B}\big(\mathscr{F}^{-1}[A_L(S)\mathscr{F}(\mathscr{R}f)(S,\theta)]\big)(x,y) = f_L(x,y)$$

is defined almost everywhere on $\mathbb{R}^2$ by Lemma 2.3, where we have $f_L \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$.

Recall that the band-limited function $q_L = \mathscr{F}^{-1}A_L$ in (9) is well-defined, where $q_L \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$. Therefore, for any fixed $\theta \in [0,\pi)$, the Fourier inversion formula

$$A_L(S) = \mathscr{F}(\mathscr{F}^{-1}A_L)(S) = \mathscr{F} q_L(S,\theta)$$

holds in the L^2-sense, in particular for almost every $S \in \mathbb{R}$.

Since $\mathscr{R}f(\cdot,\theta) \in \mathrm{L}^1(\mathbb{R})$ and $q_L(\cdot,\theta) \in \mathrm{L}^2(\mathbb{R})$, we obtain the representation

$$A_L(S)\mathscr{F}(\mathscr{R}f)(S,\theta) = \mathscr{F}(q_L * \mathscr{R}f)(S,\theta) \quad \text{for almost every } S \in \mathbb{R}$$

from the Fourier convolution theorem, Lemma 3.2. Moreover, by Young's inequality (see e.g. [3, Theorem 3.9.4]) we have $(q_L * \mathscr{R}f)(\cdot,\theta) \in \mathrm{L}^2(\mathbb{R})$, for any $\theta \in [0,\pi)$, and so the Fourier inversion formula holds again in the L^2-sense. Hence, for any $\theta \in [0,\pi)$ we obtain the representation

$$(q_L * \mathscr{R}f)(S,\theta) = \mathscr{F}^{-1}[\mathscr{F}(q_L * \mathscr{R}f)(S,\theta)] = \mathscr{F}^{-1}[A_L(S)\mathscr{F}(\mathscr{R}f)(S,\theta)]$$

for almost every $S \in \mathbb{R}$. But this implies

$$f_L = \frac{1}{2} \mathscr{B}(q_L * \mathscr{R} f) \qquad \text{a.e. on } \mathbb{R}^2,$$

where $f_L \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ due to Lemma 2.3, since $(q_L * \mathscr{R} f) \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$. □

4. Properties of the Convolution Kernel

Before we turn to error estimates on e_L, we first analyse the properties of the FBP approximation f_L, where we will show $f \in \mathrm{L}^2(\mathbb{R}^2)$. Moreover, it is convenient to express the FBP reconstruction (8) in terms of the target function f as

$$f_L = f * K_L \tag{10}$$

with the convolution kernel $K_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as in (7) given by

$$K_L(x,y) = \frac{1}{2} \mathscr{B} q_L(x,y) \qquad \text{for } (x,y) \in \mathbb{R}^2. \tag{11}$$

Since $q_L \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$, the convolution kernel K_L is defined almost everywhere on $\mathbb{R}^2$ and satisfies $K_L \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ by Lemma 2.3. In the following proposition, we prove $K_L \in \mathrm{L}^2(\mathbb{R}^2)$. Moreover, we determine the Fourier transform of K_L, as needed in the upcoming analysis on the reconstruction error. To this end, we extend the window function $W_L = W(\cdot/L)$ to $\mathbb{R}^2$ by its radialization $W_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$, i.e.,

$$W_L(x,y) = W\left(\frac{r(x,y)}{L}\right) \qquad \text{for } (x,y) \in \mathbb{R}^2, \tag{12}$$

where we let $r(x,y) = \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$.

Proposition 4.1. *Let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even with compact support* $\mathrm{supp}(W) \subseteq [-1,1]$. *Then, for any $L > 0$ the convolution kernel K_L in* (11) *satisfies $K_L \in \mathscr{C}_0(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$ and its Fourier transform is given by*

$$\mathscr{F} K_L(x,y) = W_L(x,y) \qquad \textit{for almost every } (x,y) \in \mathbb{R}^2 \tag{13}$$

with the compactly supported bivariate window function $W_L \in \mathrm{L}^\infty(\mathbb{R}^2)$ defined in (12).

Proof: Since $W \in \mathrm{L}^\infty(\mathbb{R})$ has compact support, the bivariate window function W_L in (12) is compactly supported and satisfies $W_L \in \mathrm{L}^p(\mathbb{R}^2)$ for all $1 \leq p \leq \infty$.

In particular, we have $W_L \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$. Hence, $\mathscr{F}^{-1} W_L \in \mathscr{C}_0(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$ due to the Riemann-Lebesgue lemma and the Rayleigh-Plancherel theorem. Moreover,

$$\begin{aligned}\mathscr{F}^{-1} W_L(x,y) &= \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} W_L(X,Y)\, \mathrm{e}^{\mathrm{i}(xX+yY)}\, \mathrm{d}X\, \mathrm{d}Y \\ &= \frac{1}{4\pi^2} \int_0^\pi \int_{\mathbb{R}} W(S/L)\, |S|\, \mathrm{e}^{\mathrm{i}S(x\cos(\theta)+y\sin(\theta))}\, \mathrm{d}S\, \mathrm{d}\theta \\ &= \frac{1}{4\pi^2} \int_0^\pi \int_{\mathbb{R}} A_L(S)\, \mathrm{e}^{\mathrm{i}S(x\cos(\theta)+y\sin(\theta))}\, \mathrm{d}S\, \mathrm{d}\theta\end{aligned}$$

for all $(x,y) \in \mathbb{R}^2$, by transforming from Cartesian to polar coordinates.

Recall $A_L \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$ and $q_L \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$. By Fubini's theorem, we get

$$\mathscr{F}^{-1} W_L(x,y) = \frac{1}{2\pi} \int_0^\pi q_L(x\cos(\theta)+y\sin(\theta),\theta)\, \mathrm{d}\theta = \frac{1}{2} \mathscr{B} q_L(x,y) = K_L(x,y)$$

for all $(x,y) \in \mathbb{R}^2$. Hence, we have $K_L \in \mathscr{C}_0(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$. By Fourier inversion, we finally get the identity (13) in the L^2-sense, in particular almost everywhere on $\mathbb{R}^2$. □

Before we proceed, we wish to add one more remark concerning the convolution kernel K_L. To this end, note that the bivariate window function W_L in (12) has compact support. Therefore, K_L is analytic, due to the Paley-Wiener theorem, i.e., $K_L \in \mathscr{C}^\infty(\mathbb{R}^2)$.

We are now in a position where we can prove the desired representation in (10), i.e.,

$$f_L = \frac{1}{2} \mathscr{B}(q_L * \mathscr{R} f) = f * K_L \tag{14}$$

in the L^2-sense. In particular, we can show that $f_L \in \mathrm{L}^2(\mathbb{R}^2)$.

Proposition 4.2. *Let $f \in \mathrm{L}^1(\mathbb{R}^2)$. Moreover, let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even and compactly supported with* $\mathrm{supp}(W) \subseteq [-1,1]$. *Then, $f_L \in \mathrm{L}^2(\mathbb{R}^2)$, for any $L > 0$, where we have*

$$f_L = f * K_L$$

in the L^2-sense, in particular almost everywhere on $\mathbb{R}^2$.

Proof: Since $f \in \mathrm{L}^1(\mathbb{R}^2)$ by assumption and $K_L \in \mathrm{L}^2(\mathbb{R}^2)$ due to Proposition 4.1, Young's inequality yields $f * K_L \in \mathrm{L}^2(\mathbb{R}^2)$. Therefore, the Fourier inversion formula

$$(f * K_L)(x,y) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathscr{F}(f * K_L)(X,Y)\, \mathrm{e}^{\mathrm{i}(xX+yY)}\, \mathrm{d}X\, \mathrm{d}Y$$

holds in the L^2-sense, in particular for almost every $(x,y) \in \mathbb{R}^2$.

From the Fourier convolution theorem, Lemma 3.2, and Proposition 4.1, we get

$$\mathscr{F}(f * K_L) = \mathscr{F} f \cdot \mathscr{F} K_L = W_L \cdot \mathscr{F} f \quad \text{a.e. on } \mathbb{R}^2,$$

which in turn implies

$$(f * K_L)(x,y) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathscr{F} f(X,Y) \, W_L(X,Y) \, \mathrm{e}^{\mathrm{i}(xX+yY)} \, \mathrm{d}X \, \mathrm{d}Y.$$

Since $W \in \mathrm{L}^\infty(\mathbb{R})$ has compact support and $f \in \mathrm{L}^1(\mathbb{R}^2)$, we can apply Fubini's theorem to obtain

$$(f * K_L)(x,y) = \frac{1}{4\pi^2} \int_0^\pi \int_{\mathbb{R}} \mathscr{F} f(S\cos(\theta), S\sin(\theta)) \, W(S/L) \, |S| \, \mathrm{e}^{\mathrm{i}S(x\cos(\theta)+y\sin(\theta))} \, \mathrm{d}S \, \mathrm{d}\theta$$

by transformation $(X,Y) = (S\cos(\theta), S\sin(\theta))$ from Cartesian to polar coordinates.

Now, for $f \in \mathrm{L}^1(\mathbb{R}^2)$, the *Fourier slice theorem* (see e.g. [9, Theorem II.1.1]) yields

$$\mathscr{F} f(S\cos(\theta), S\sin(\theta)) = \mathscr{F}(\mathscr{R} f)(S,\theta) \quad \text{for all } (S,\theta) \in \mathbb{R} \times [0,\pi),$$

which in turn implies

$$\begin{aligned}(f * K_L)(x,y) &= \frac{1}{4\pi^2} \int_0^\pi \int_{\mathbb{R}} \mathscr{F}(\mathscr{R} f)(S,\theta) \, W(S/L) \, |S| \, \mathrm{e}^{\mathrm{i}S(x\cos(\theta)+y\sin(\theta))} \, \mathrm{d}S \, \mathrm{d}\theta \\ &= \frac{1}{4\pi^2} \int_0^\pi \int_{\mathbb{R}} \mathscr{F}(\mathscr{R} f)(S,\theta) \, A_L(S) \, \mathrm{e}^{\mathrm{i}S(x\cos(\theta)+y\sin(\theta))} \, \mathrm{d}S \, \mathrm{d}\theta.\end{aligned}$$

Since A_L is in $\mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$, we have $\mathscr{F}^{-1} A_L \in \mathrm{L}^2(\mathbb{R})$ and so

$$A_L(S) = \mathscr{F}(\mathscr{F}^{-1} A_L)(S)$$

holds in the L^2-sense, in particular for almost every $S \in \mathbb{R}$. Therefore, by Young's inequality we have

$$(q_L * \mathscr{R} f)(\cdot,\theta) \in \mathrm{L}^2(\mathbb{R}) \quad \text{for all } \theta \in [0,\pi)$$

for the band-limited function $q_L \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$ in (9). Moreover, by the Fourier convolution theorem, Lemma 3.2, the identity

$$\mathscr{F}(q_L * \mathscr{R} f)(S,\theta) = \mathscr{F} q_L(S,\theta) \, \mathscr{F}(\mathscr{R} f)(S,\theta) = A_L(S) \, \mathscr{F}(\mathscr{R} f)(S,\theta)$$

holds in the L^2-sense, for any $\theta \in [0,\pi)$, since $\mathscr{R}f(\cdot,\theta) \in \mathrm{L}^1(\mathbb{R})$ and $q_L(\cdot,\theta) \in \mathrm{L}^2(\mathbb{R})$. Therefore, we obtain

$$(f * K_L)(x,y) = \frac{1}{4\pi^2} \int_0^\pi \int_{\mathbb{R}} \mathscr{F}(q_L * \mathscr{R}f)(S,\theta)\, \mathrm{e}^{\mathrm{i}S(x\cos(\theta)+y\sin(\theta))}\, \mathrm{d}S\, \mathrm{d}\theta$$
$$= \frac{1}{2\pi} \int_0^\pi \mathscr{F}^{-1}[\mathscr{F}(q_L * \mathscr{R}f)](x\cos(\theta)+y\sin(\theta),\theta)\, \mathrm{d}\theta.$$

Since $(q_L * \mathscr{R}f)(\cdot,\theta) \in \mathrm{L}^2(\mathbb{R})$, for any $\theta \in [0,\pi)$, the Fourier inversion formula yields

$$(q_L * \mathscr{R}f)(S,\theta) = \mathscr{F}^{-1}[\mathscr{F}(q_L * \mathscr{R}f)](S,\theta) \quad \text{for almost every } S \in \mathbb{R}.$$

This finally implies $f_L \in \mathrm{L}^2(\mathbb{R}^2)$, along with the stated representation, since

$$\begin{aligned}(f * K_L)(x,y) &= \frac{1}{2\pi} \int_0^\pi (q_L * \mathscr{R}f)(x\cos(\theta)+y\sin(\theta),\theta)\, \mathrm{d}\theta \\ &= \frac{1}{2} \mathscr{B}(q_L * \mathscr{R}f)(x,y) = f_L(x,y)\end{aligned}$$

holds in the L^2-sense, in particular for almost every $(x,y) \in \mathbb{R}^2$. □

5. Analysis of the FBP Reconstruction Error

Now we analyse the reconstruction error $e_L = f - f_L$ of the FBP approximation f_L for target functions $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, where $\alpha > 0$. To this end, we prove Sobolev error estimates and convergence rates for e_L with respect to the $\mathrm{H}^\sigma(\mathbb{R}^2)$-norm, for $0 \leq \sigma \leq \alpha$. This in particular gives L^2-error estimates, when $\sigma = 0$.

Throughout this section we assume that the low-pass filter's window $W \in \mathrm{L}^\infty(\mathbb{R})$ is even and compactly supported with $\mathrm{supp}(W) \subseteq [-1,1]$. This is in contrast to our previous work [1, 2], where we implicitly required the more restrictive condition $W \in \mathscr{C}(\mathbb{R})$.

5.1. L^2*-Error Estimates*

Let us now turn to the analysis of the reconstruction error $e_L = f - f_L$ in $\mathrm{L}^2(\mathbb{R}^2)$. To start with, we assume $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$. By Proposition 4.2, under the stated assumptions on f and W, we have $f_L \in \mathrm{L}^2(\mathbb{R}^2)$ for the FBP approximation, along with the representation (14), i.e., $f_L = f * K_L$. But this immediately implies

$$\|f - f_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \|f - f * K_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \frac{1}{4\pi^2} \|\mathscr{F}f - \mathscr{F}f \cdot \mathscr{F}K_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)}$$

by the Fourier convolution formula, Lemma 3.2, and the Rayleigh-Plancherel theorem. Moreover, by Proposition 4.1, the Fourier transform $\mathscr{F}K_L$ of the convolution kernel K_L in (11) is given by the bivariate window function W_L in (12), i.e., $\mathscr{F}K_L = W_L$, so that

$$\begin{aligned}\|f - f_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} &= \frac{1}{4\pi^2}\,\|\mathscr{F}f - W_L \cdot \mathscr{F}f\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} |(\mathscr{F}f - W_L \cdot \mathscr{F}f)(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Now we split this representation for $\|f - f_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)}$ into a sum of two integrals,

$$\|f - f_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = I_1 + I_2,$$

where we let

$$\begin{aligned}I_1 &= \frac{1}{4\pi^2}\int_{\left\|\binom{x}{y}\right\|_2 \le L} |(\mathscr{F}f - W_L \cdot \mathscr{F}f)(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y, \\ I_2 &= \frac{1}{4\pi^2}\int_{\left\|\binom{x}{y}\right\|_2 > L} |\mathscr{F}f(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We analyse the two error terms separately. Integral I_1 can be bounded above by

$$\begin{aligned}I_1 &= \frac{1}{4\pi^2}\int_{r(x,y)\le L} \big(1 - W_L(r(x,y))\big)^2\,|\mathscr{F}f(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\le \|1 - W_L\|^2_{\infty,[-L,L]}\,\frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} |\mathscr{F}f(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y = \|1 - W\|^2_{\infty,[-1,1]}\,\|f\|^2_{\mathrm{L}^2(\mathbb{R}^2)},\end{aligned}$$

whereas for $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$, $\alpha > 0$, integral I_2 can be bounded above by

$$\begin{aligned}I_2 &= \frac{1}{4\pi^2}\int_{r(x,y)>L} \big(1 + x^2 + y^2\big)^{\alpha}\big(1 + x^2 + y^2\big)^{-\alpha}\,|\mathscr{F}f(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{1}{4\pi^2}\int_{r(x,y)>L} \big(1 + x^2 + y^2\big)^{\alpha} L^{-2\alpha}\,|\mathscr{F}f(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\le L^{-2\alpha}\,\|f\|^2_\alpha.\end{aligned}$$

We can summarize our discussion as follows.

Theorem 5.1 (L^2**-error estimate).** *For $\alpha > 0$ let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$. Moreover, let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even and compactly supported with* $\mathrm{supp}(W) \subseteq [-1,1]$. *Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is for any $L > 0$ bounded above by*

$$\|f - f_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \le \|1 - W\|_{\infty,[-1,1]}\,\|f\|_{\mathrm{L}^2(\mathbb{R}^2)} + L^{-\alpha}\,\|f\|_\alpha. \tag{15}$$

□

Before we proceed, we wish to make a few remarks about the result of Theorem 5.1.

First note that the first term on the right hand side of (15) depends on W and f, but not on L. To obtain convergence $\|f - f_L\|_{\mathrm{L}(\mathbb{R}^2)} \longrightarrow 0$ for $L \longrightarrow \infty$ from (15), we require $\|1-W\|_{\infty,[-1,1]} = 0$, which is satisfied by the window $W = \chi_{[-1,1]}$ of the *Ram-Lak filter*. Moreover, for $W = \chi_{[-1,1]}$, the smoothness α of f determines the decay rate in (15) by

$$L^{-\alpha}\,\|f\|_\alpha = \mathscr{O}(L^{-\alpha}) \quad \text{for } L \longrightarrow \infty.$$

We remark that the L^2-error estimate (15) in Theorem 5.1 was first presented in [1], but under stronger assumptions. In fact, in [1, Theorem 1] we assume $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ for the convolution kernel, in which case the window function $W_L = \mathscr{F}K_L$ in (12) is continuous, i.e., $W_L \in \mathscr{C}(\mathbb{R}^2)$, due to the Riemann-Lebesgue lemma. But this assumption on W_L, i.e. on W, is rather restrictive and, in particular, *not* satisfied by commonly used window functions W, e.g. those of the filters Ram-Lak or Shepp-Logan (cf. Table 1).

5.2. *Sobolev Error Estimates*

In this subsection, we prove H^σ-Sobolev error estimates, for $0 \le \sigma \le \alpha$, for the FBP reconstruction error e_L for target functions $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$.

We first show that the FBP approximation f_L belongs to the Sobolev space $\mathrm{H}^\sigma(\mathbb{R}^2)$ for $0 \le \sigma \le \alpha$. To this end, recall that we analysed the convolution kernel K_L by Proposition 3.1, where we found $K_L \in \mathscr{C}_0(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$ and, moreover, $\mathscr{F}K_L = W_L$. This in combination with the representation $f_L = f * K_L$ in Proposition 4.2 yields

$$\begin{aligned}
\|f_L\|_\sigma^2 &= \|f * K_L\|_\sigma^2 \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \left(1 + r(x,y)^2\right)^\sigma |(W_L \cdot \mathscr{F}f)(x,y)|^2 \,\mathrm{d}(x,y) \\
&\le \Big(\sup_{r(x,y)\le L} |W_L(x,y)|^2\Big)\,\|f\|_\alpha^2 = \|W\|_{\infty,[-1,1]}^2\,\|f\|_\alpha^2.
\end{aligned}$$

Therefore, we have $f_L \in \mathrm{H}^\sigma(\mathbb{R}^2)$ for any $0 \le \sigma \le \alpha$.

Let us now analyse the FBP reconstruction error e_L with respect to the H^σ-norm. For $\gamma \ge 0$, we define

$$r_\gamma(x,y) = \left(1 + r(x,y)^2\right)^\gamma = \left(1 + x^2 + y^2\right)^\gamma \quad \text{for } (x,y) \in \mathbb{R}^2$$

so that the H^σ-norm of $e_L = f - f_L$ can be expressed as

$$
\begin{aligned}
\|f - f_L\|_\sigma^2 &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} r_\sigma(x,y)\,|\mathscr{F}(f - f_L)(x,y)|^2\,\mathrm{d}(x,y) \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} r_\sigma(x,y)\,|(\mathscr{F}f - W_L \cdot \mathscr{F}f)(x,y)|^2\,\mathrm{d}(x,y) \\
&= I_1 + I_2,
\end{aligned}
$$

where for $B_L = \left\{(x,y) \in \mathbb{R}^2 \mid r(x,y) \leq L\right\}$ we let

$$
I_1 = \frac{1}{4\pi^2}\int_{B_L} r_\sigma(x,y)\,|1 - W_L(x,y)|^2\,|\mathscr{F}f(x,y)|^2\,\mathrm{d}(x,y) \tag{16}
$$

$$
I_2 = \frac{1}{4\pi^2}\int_{\mathbb{R}^2\setminus B_L} r_\sigma(x,y)\,|\mathscr{F}f(x,y)|^2\,\mathrm{d}(x,y). \tag{17}
$$

Now for $\gamma \geq 0$, we define the function

$$
\Phi_{\gamma,W}(L) = \sup_{S\in[-1,1]} \frac{(1 - W(S))^2}{(1 + L^2S^2)^\gamma} \quad \text{for } L > 0,
$$

as already mentioned in (5), so that we can bound the integral I_1 in (16) from above by

$$
I_1 \leq \left(\sup_{(x,y)\in B_L} \frac{(1 - W_L(x,y))^2}{r_{\alpha-\sigma}(x,y)}\right)\|f\|_\alpha^2 = \Phi_{\alpha-\sigma,W}(L)\|f\|_\alpha^2,
$$

where we used the identity

$$
\sup_{(x,y)\in B_L} \frac{(1 - W_L(x,y))^2}{r_{\alpha-\sigma}(x,y)} = \sup_{S\in[-L,L]} \frac{(1 - W(S/L))^2}{(1 + S^2)^{\alpha-\sigma}}.
$$

For $0 \leq \sigma \leq \alpha$, we can bound the integral I_2 in (17) by

$$
\begin{aligned}
I_2 &\leq L^{2(\sigma-\alpha)}\,\frac{1}{4\pi^2}\int_{\mathbb{R}^2\setminus B_L} r_\sigma(x,y)\,|\mathscr{F}f(x,y)|^2\,\mathrm{d}(x,y) \\
&\leq L^{2(\sigma-\alpha)}\,\|f\|_\alpha^2.
\end{aligned}
$$

Combining the estimates for I_1 and I_2, we finally obtain

$$
\|f - f_L\|_\sigma^2 \leq \left(\Phi_{\alpha-\sigma,W}(L) + L^{2(\sigma-\alpha)}\right)\|f\|_\alpha^2,
$$

so that we can summarize the discussion of this subsection as follows.

Theorem 5.2 (**H^σ-error estimate**). *For $\alpha > 0$, let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$. Moreover, let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even and compactly supported with* $\operatorname{supp}(W) \subseteq [-1,1]$. *Then, for $0 \leq \sigma \leq \alpha$, the H^σ-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$
\|f - f_L\|_\sigma \leq \left(\Phi_{\alpha-\sigma,W}^{1/2}(L) + L^{\sigma-\alpha}\right)\|f\|_\alpha. \tag{18}
$$

For $\sigma = 0$ we obtain an L^2-error estimate which is more refined than that in (15).

Corollary 5.3 (Refined L^2-error estimate). *Suppose $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for $\alpha > 0$. Moreover, let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even and compactly supported with* $\mathrm{supp}(W) \subseteq [-1,1]$. *Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$\|f - f_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\Phi_{\alpha,W}^{1/2}(L) + L^{-\alpha}\right) \|f\|_\alpha. \tag{19}$$

□

We remark that the more refined L^2-error estimate in (19) agrees with that of our previous result in [2, Theorem 4.1], where, in contrast to Corollary 5.3, we required the more restrictive condition $K_L \in \mathrm{L}^1(\mathbb{R}^2)$, and so $W \in \mathscr{C}(\mathbb{R})$.

Like in the L^2-error estimate of [2, Theorem 4.1], the H^σ-error estimate in (18) and the L^2-error estimate in (19) involves the error term $\Phi_{\gamma,W}(L)$, for $\gamma = \alpha - \sigma$ in (18) and for $\gamma = \alpha$ in (19). Therefore, we can rely on the analysis in [2] concerning the properties of $\Phi_{\gamma,W}(L)$.

5.3. Convergence Rates in Sobolev Spaces

In this section, we analyse the convergence of the reconstruction error $e_L = f - f_L$ in (4). To this end, let $S^*_{\gamma,W,L} \in [0,1]$, for $\gamma \geq 0$, be the smallest maximizer in $[0,1]$ of the even function

$$\Phi_{\gamma,W,L}(S) = \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\gamma} \quad \text{for } S \in [-1,1].$$

To determine the rate of convergence for $\|f - f_L\|_\sigma$, we assume that $S^*_{\alpha-\sigma,W,L}$ is uniformly bounded away from 0, i.e., there exists a constant $c_{\alpha-\sigma,W} > 0$ satisfying

$$S^*_{\alpha-\sigma,W,L} \geq c_{\alpha-\sigma,W} \qquad \text{for all } L > 0. \tag{20}$$

Then, the error term $\Phi_{\alpha-\sigma,W}(L)$ is bounded above by

$$\Phi_{\alpha-\sigma,W}(L) = \frac{\left(1 - W(S^*_{\alpha-\sigma,W,L})\right)^2}{\left(1 + L^2 (S^*_{\alpha-\sigma,W,L})^2\right)^{\alpha-\sigma}} \leq c_{\alpha-\sigma,W}^{2(\sigma-\alpha)} \|1 - W\|^2_{\infty,[-1,1]} L^{2(\sigma-\alpha)}.$$

In this case, we obtain

$$\|f - f_L\|_\sigma^2 \leq \left(c_{\alpha-\sigma,W}^{2(\sigma-\alpha)} \|1 - W\|^2_{\infty,[-1,1]} + 1\right) L^{2(\sigma-\alpha)} \|f\|_\alpha^2 = \mathscr{O}(L^{2(\sigma-\alpha)})$$

for $L \longrightarrow \infty$.

In summary, this yields the following result.

Theorem 5.4 (Rate of convergence in H^σ). *Let the assumptions of Theorem 5.2 and in* (20) *be satisfied. Then, for* $0 \leq \sigma \leq \alpha$*, the* H^σ*-norm of the FBP reconstruction error* $e_L = f - f_L$ *is bounded above by*

$$\|f - f_L\|_\sigma \leq \left(c_{\alpha-\sigma,W}^{\sigma-\alpha} \, \|1 - W\|_{\infty,[-1,1]} + 1 \right) L^{\sigma-\alpha} \, \|f\|_\alpha = \mathscr{O}(L^{\sigma-\alpha}) \tag{21}$$

for $L \longrightarrow \infty$. □

Note that the decay rate $\alpha - \sigma$ in (21) is determined by the difference between the smoothness α of the target function f and the order σ of the Sobolev norm in which the reconstruction error e_L is measured. Moreover, note that for the special case $\sigma = 0$ we obtain the following L^2-error estimate.

Corollary 5.5 (Rate of convergence in L^2). *Let the assumptions of Theorem 5.2 and in* (20) *be satisfied. Then, the* L^2*-norm of the FBP reconstruction error* $e_L = f - f_L$ *is bounded above by*

$$\|f - f_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(c_{\alpha,W}^{-\alpha} \, \|1 - W\|_{\infty,[-1,1]} + 1 \right) L^{-\alpha} \, \|f\|_\alpha = \mathscr{O}(L^{-\alpha})$$

for $L \longrightarrow \infty$. □

We remark that assumption (20) is satisfied for a large class of window functions. For example, let W satisfy $W(S) = 1$, for all $S \in (-\varepsilon, \varepsilon)$, with $\varepsilon \in (0,1)$. Then, assumption (20) is fulfilled with the constant $c_{\alpha-\sigma,W} = \varepsilon$ for all $0 \leq \sigma \leq \alpha$.

6. Conclusion

We have proven L^2 and Sobolev error estimates, along with convergence rates, for filtered back projection to approximate target functions f from fractional Sobolev spaces $\mathrm{H}^\alpha(\mathbb{R}^2)$. We only require $W \in \mathrm{L}^\infty(\mathbb{R})$, with $\mathrm{supp}(W) \subseteq [-1,1]$, for the low-pass filter's window W. This is in contrast to our previous work [1, 2], where we relied on the more restrictive condition $W \in \mathscr{C}(\mathbb{R})$ to obtain error estimates and convergence rates in $\mathrm{L}^2(\mathbb{R}^2)$. Therefore, the analysis of this paper covers a larger class of low-pass filters, including the Ram-Lak und the Shepp-Logan filter, and, moreover, generalises our previous error estimates from $\mathrm{L}^2(\mathbb{R}^2)$ to Sobolev spaces $\mathrm{H}^\sigma(\mathbb{R}^2)$, for $0 \leq \sigma \leq \alpha$.

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