On the error behaviour of the filtered back projection

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The filtered back projection (FBP) formula allows us to reconstruct bivariate functions from given Radon samples. However, the FBP formula is numerically unstable and low-pass filters with finite bandwidth and a compactly supported window function are employed to make the reconstruction by FBP less sensitive to noise. In this paper we analyse the inherent reconstruction error which is incurred by the application of a low-pass filter with finite bandwidth. We present L^2-error estimates on Sobolev spaces of fractional order along with asymptotic convergence rates, where the filter's bandwidth goes to infinity.

1 Filtered back projection

The term *filtered back projection* (FBP) refers to a well-known and commonly used reconstruction technique in computerized tomography (CT). The classical CT reconstruction problem can be formulated as follows.

Problem 1.1 Let $\Omega \subset \mathbb{R}^2$ be bounded. Reconstruct a bivariate function f with compact support $\operatorname{supp}(f) \subseteq \Omega$ from its given Radon data

$$\{\mathcal{R}f(t,\theta) \mid t \in \mathbb{R},\, \theta \in [0,\pi)\},$$

where the *Radon transform* $\mathcal{R}f$ of $f \in \mathrm{L}^1(\mathbb{R}^2)$ is defined as

$$\mathcal{R}f(t,\theta) = \int_{\{x\cos(\theta)+y\sin(\theta)=t\}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for $(t,\theta) \in \mathbb{R} \times [0,\pi)$.

Thus, the CT reconstruction problem seeks for the inversion of the Radon transform $\mathcal{R}$. For a comprehensive mathematical treatment of $\mathcal{R}$ and its inversion, we refer to [5, 11].

The inversion of $\mathcal{R}$ is well understood and given by the classical *filtered back projection formula* ([4, Theorem 6.2.])

$$f(x,y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}[|S|\mathcal{F}(\mathcal{R}f)(S,\theta)]\big)(x,y), \qquad (1)$$

which holds for $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{C}(\mathbb{R}^2)$, and where the *back projection* $\mathcal{B}h$ of $h \in \mathrm{L}^1(\mathbb{R} \times [0,\pi))$ is defined as

$$\mathcal{B}h(x,y) = \frac{1}{\pi}\int_0^\pi h(x\cos(\theta)+y\sin(\theta),\theta)\,\mathrm{d}\theta$$

for $(x,y) \in \mathbb{R}^2$. Note that $\mathcal{B}$ is the adjoint operator of $\mathcal{R}$.

We remark that the FBP formula (1) is highly sensitive with respect to noise and, consequently, numerically *unstable*. To stabilize it, we follow a standard approach and replace the factor $|S|$ in (1) by a *low-pass filter* A_L of the form

$$A_L(S) = |S|W({S}/{L})$$

with finite *bandwidth* $L > 0$ and an even *window function* $W \in \mathrm{L}^\infty(\mathbb{R})$ with compact support $\operatorname{supp}(W) \subseteq [-1,1]$.

By applying the low-pass filter $A_L(S)$, the reconstruction of f is no longer exact, but the resulting *approximate FBP reconstruction* f_L can be rewritten as

$$f_L = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L * \mathcal{R}f\big) = f * K_L,$$

where we define the *convolution kernel* $K_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ via

$$K_L(x,y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L\big)(x,y).$$

2 Error analysis

In the following, we analyse the intrinsic FBP reconstruction error $e_L = f - f_L$ which is incurred by the application of the low-pass filter A_L. We remark at this point that pointwise and L^∞-error estimates on e_L were proven by Munshi et al. in [7, 8], supported by numerical experiments in [9]. Error bounds on the L^p-norm of e_L, in terms of an L^p-modulus of continuity of f, were proven by Madych in [6].

Our goal is to proof L^2-error estimates on e_L for the relevant case of target functions f from Sobolev spaces of fractional order (cf. [10]), i.e., we assume

$$f \in \mathrm{H}^\alpha(\mathbb{R}^2) = \big\{g \in \mathcal{S}'(\mathbb{R}^2) \mid \|g\|_\alpha < \infty\big\} \quad \text{for } \alpha > 0,$$

where

$$\|g\|_\alpha^2 = \frac{1}{2\pi}\int_\mathbb{R}\int_\mathbb{R}(1+x^2+y^2)^\alpha\,|\mathcal{F}g(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y.$$

The presented results are refinements and extensions of our results in [2,3]. More details and proofs, along with numerical simulations, can be found in [1].

Theorem 2.1 *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for $\alpha > 0$, $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ and $W \in \mathcal{C}([-1,1])$ with $W(0) = 1$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \le \Big(\Phi_{\alpha,W}^{1/2}(L) + L^{-\alpha}\Big)\,\|f\|_\alpha,$$

where

$$\Phi_{\alpha,W}(L) = \max_{S\in[-1,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \longrightarrow 0 \quad \textit{for} \quad L \longrightarrow \infty.$$

In particular,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \longrightarrow 0 \quad \textit{for} \quad L \longrightarrow \infty.$$

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3 Rate of convergence

We now analyse the convergence rate of the FBP reconstruction error $\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)}$ as L goes to ∞. To this end, let $S^*_{\alpha,W,L}$ denote the smallest maximizer in $[0,1]$ of the even function

$$\Phi_{\alpha,W,L}(S) = \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \quad \text{for } S \in [-1,1].$$

Our further analysis relies on the following assumption.

Assumption 3.1 $S^*_{\alpha,W,L}$ is uniformly bounded away from zero, i.e., there exists a constant $c_{\alpha,W} > 0$ such that

$$S^*_{\alpha,W,L} \geq c_{\alpha,W} \quad \forall\, L > 0.$$

Under this assumption, we can proof L^2-error estimates for the FBP reconstruction with convergence rates as follows.

Theorem 3.2 *Under Assumption 3.1 and the assumptions of Theorem 2.1, the* L^2*-norm of* e_L *is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(c_{\alpha,W}^{-\alpha}\, \|1-W\|_{\infty,[-1,1]} + 1\right) L^{-\alpha}\, \|f\|_\alpha.$$

In particular,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}(L^{-\alpha}) \quad \textit{for} \quad L \longrightarrow \infty.$$

Note that the decay rate of the L^2-error in Theorem 3.2 is determined by the smoothness α of the target function f.

We remark that Assumption 3.1 is satisfied for a large class of window functions. For example, let $W \in \mathcal{C}([-1,1])$ satisfy

$$W(S) = 1 \quad \forall\, S \in [-\varepsilon,\varepsilon] \qquad \text{for some } \varepsilon \in (0,1).$$

Then, Assumption 3.1 is fulfilled with $c_{\alpha,W} = \varepsilon$.

But Assumption 3.1 is not satisfied for *all* commonly used choices of W. In fact, in [1] we investigated the behaviour of $S^*_{\alpha,W,L}$ and $\Phi_{\alpha,W}$ numerically for the following window functions of the filter $A_L(S) = |S|\, W(S/L)$:

Name	$W(S)$ for $\lvert S\rvert \leq 1$	Parameter
Shepp–Logan	$\mathrm{sinc}(\pi S/2)$	-
Cosine	$\cos(\pi S/2)$	-
Hamming	$\beta + (1-\beta)\cos(\pi S)$	$\beta \in [1/2, 1]$
Gaussian	$\exp(-(\pi S/\beta)^2)$	$\beta > 1$

We summarize our numerical results from [1] as follows. For $\alpha < 2$, we found that Assumption 3.1 is fulfilled and

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

For $\alpha \geq 2$, Assumption 3.1 is not fulfilled, since

$$S^*_{\alpha,W,L} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty,$$

and the convergence rate of $\Phi_{\alpha,W}$ stagnates at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-4}) \quad \text{for} \quad L \longrightarrow \infty.$$

To show the observed behaviour of $\Phi_{\alpha,W}(L)$, we note that all above windows W are in $\mathcal{C}^2([-1,1])$ with $W(0) = 1$ and $W'(0) = 0$. With assuming $W \in \mathcal{C}^k([-1,1])$, for $k \geq 2$, we can proof the following estimate on $\Phi_{\alpha,W}(L)$.

Theorem 3.3 *Let* $W \in \mathcal{C}^k([-1,1])$*, for* $k \geq 2$*, with*

$$W(0) = 1 \quad \textit{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Moreover, let $\alpha > 0$*. Then,* $\Phi_{\alpha,W}(L)$ *is bounded above by*

$$\Phi_{\alpha,W}(L) \leq \begin{cases} \frac{c^2_{\alpha,k}}{(k!)^2}\, \|W^{(k)}\|^2_{\infty,[-1,1]}\, L^{-2k} & \textit{for } \alpha > k \\ \frac{1}{(k!)^2}\, \|W^{(k)}\|^2_{\infty,[-1,1]}\, L^{-2\alpha} & \textit{for } \alpha \leq k \end{cases}$$

for sufficiently large $L > 0$ *and with the constant*

$$c_{\alpha,k} = \left(\frac{k}{\alpha-k}\right)^{k/2} \left(\frac{\alpha-k}{\alpha}\right)^{\alpha/2} \quad \textit{for } \alpha > k.$$

In particular,

$$\Phi_{\alpha,W}(L) = \mathcal{O}\left(L^{-2\min\{k,\alpha\}}\right) \quad \textit{for} \quad L \longrightarrow \infty.$$

The theoretical results of Theorem 3.3 comply with the above observations, where $k = 2$, as well as with the numerical results for arbitrary $k \geq 2$ in [1]. In particular, the saturation of the convergence rate at $\mathcal{O}(L^{-2k})$ was observed. Therefore, our numerical experiments show that the proven convergence rate of $\Phi_{\alpha,W}(L)$ is optimal for $\mathcal{C}^k$-windows.

Combining Theorems 2.1 and 3.3, we finally obtain the following result for the FBP reconstruction with $\mathcal{C}^k$-windows.

Corollary 3.4 *Let the assumptions of Theorems 2.1 and 3.3 be satisfied. Then, the* L^2*-norm of* e_L *is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\frac{1}{k!}\, \|W^{(k)}\|_{\infty,[-1,1]} + 1\right) L^{-\alpha}\, \|f\|_\alpha$$

for $\alpha \leq k$*, and by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\frac{c_{\alpha,k}}{k!}\, \|W^{(k)}\|_{\infty,[-1,1]}\, L^{-k} + L^{-\alpha}\right) \|f\|_\alpha$$

for $\alpha > k$ *and sufficiently large* $L > 0$*. In particular,*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}\left(L^{-\min\{k,\alpha\}}\right) \quad \textit{for} \quad L \longrightarrow \infty.$$

Note that in Corollary 3.4 the decay rate of $\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)}$ is for $\alpha \leq k$ determined by the smoothness α of the target function f, whereas for $\alpha > k$ the decay rate saturates at $\mathcal{O}(L^{-k})$. Here, k denotes the differentiability order of the window W, whose first $k-1$ derivatives are required to vanish at zero.

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